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Quaternions as Spherical Particles in the Three-dimensional Conformal Space

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Abstract. The triangle model of quaternions has its origin in conformal geometry. A quaternion is a centred and directed tetraglobe in the 3-dimensional conformal space. The multiplication of quaternions is the composition of such spherical particles. Quaternions (not points or vectors) are the basic elements of natural space. Complex Hamiltonian numbers are pure angle entities. Only if a unit of length is defined these numbers can be used to describe the time and space components of physical events. The spin of quaternionic particles is the logarithm of its charge.

Introduction

In [6] and [7] hereafter referred to as I and II respectively, complex numbers were used to *describe* entities of conformal geometry (angles, conformal triangles etc.). In this third article we will use elements of conformal geometry to construct complex numbers as geometrical entities.

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For brevity I.x will denote item numbered x in I (and II.y will denote item numbered y in II). Thus Equation (I.1.2), for example, is Equation (1.2) in I; and, for instance, Theorem II.2.3 is Theorem 2.3 in II.

1. Trigons

A tetraglobe (Definition II.2.1) is always associated with a family of 2×3 numbers. This family emerges by permuting the four points which define the characteristic number (Definition II.3.1; compare also Section II.8). These six numbers can be generated by permuting the three angles which define the parametric descriptions of a tetraglobe's number family (Equations II.1.3 of Theorem II.1.1). Every fixed permutation $(\alpha_1, \alpha_2, \alpha_3)$ of the three angles is associated with one and only one of these complex numbers. We get a one-to-one correspondence between figure and number, if every tetraglobe with fixed permutation of its angles is seen as a different figure.

Definition 1.1. *A tetraglobe with a fixed permutation of its angles is a **trigon**.*

Definition 1.2. *If an angle $\pm\alpha_k$ is the parameter in $e^{\boldsymbol{i}(\pm\alpha)}$ (Equations II.1.3) α_k is called **exponent angle*** $(k = 1, 2, 3)$.

Definition 1.3. *A tetraglobe with a fixed exponent angle is a **centred tetraglobe**.*

A tetraglobe has two directions (Definition II.4.1) and we have three possibilities to give a tetraglobe a centre. A trigon is a directed and centred tetraglobe.

Every plane (Gauss plane) with the position $\boldsymbol{i}$ in the 3-dimensional space and every sphere (Riemann sphere) tangential to this plane can be used to construct a picture of a trigon. I call a conformal sphere, which is used to realise a trigon, the stage $\mathbb{C}_{\boldsymbol{i}}$ of this trigon.

Definition 1.4. *A trigon together with the position* $\boldsymbol{i}$ *of its stage* $\mathbb{C}_{\boldsymbol{i}}$ *in the 3-dimensional space is a* ***spherical particle***. *(Compare Section II.4, Remark 3)*

The four corners of a degenerate tetraglobe lie on one circle. The characteristic number of such tetra is a real number (Theorem I.5.3). Therefore also a degenerate trigon (spherical particle) represents a real number.

Remarks

1. The name 'trigon' (compare 'trigon(o)metry') is chosen because a centred and directed tetraglobe ('3-angle') is a system of three angles.

   'Particle' reminds that a conformal tetraglobe does not possess a diameter. (This spherical figure can be seen both small ('corpuscle') and big ('wave element'), compare remark in Section II.8)

2. Our way to use tetras with fixed centre and direction should be seen by analogy with the use of directed pairs of points as vectors. If a line z_1z_2 gets directions we have two vectors z_1z_2 and z_2z_1. If a tetraglobe is given centres and directions we get 3×2 trigons.

3. An angle (not a point) defines a 'centre'. Only if a trigon in special Euclidean location is drawn with the help of a Gauss plane we can associate 'centre' with the apex of the Euclidean exponent angle.

2. Trigons in Euclidean location

In Euclidean location a tetraglobe appears as an Euclidean triangle (together with its circumcircle) if we illustrate this conformal entity in a Gauss plane. In this location also a trigon is an Euclidean triangle. If, for example, the four points z_k are the corners of a trigon T_1 with $z_4 = \infty$, the points z_1, z_2, z_3 are the corners of this Euclidean triangle. If α_1 (with the apex z_1) is the exponent angle the trigon T_1 can also be seen as a pair of vectors $(+c, -b)$ with

$$+c := z_1 z_2\ , \quad -b := z_1 z_3\ . \tag{2.1}$$

If the triangle is centred by the exponent angle α_2 we may see the trigon T_2 represented by the pair of vectors

$$+a := z_2 z_3\ , \quad -c := z_2 z_1; \tag{2.2}$$

and T_3 with the exponent angle α_3 is the pair

$$+b := z_3 z_1\ , \quad -a := z_3 z_2\ . \tag{2.3}$$

The directions of the triangle's vector sides a, b, c describe the positive direction of the trigons $T_k,\ \ k = 1, 2, 3$.

In addition a negative cycle $T_{-k},\ k = 1, 2, 3$, exists represented by pairs of vectors with inverse directions:

$$\begin{array}{lll} T_{-1} & \text{is represented by} & -c := z_2 z_1, \quad +b := z_3 z_1. \\ T_{-2} & \text{is represented by} & -a := z_3 z_2, \quad +c := z_1 z_2\ . \\ T_{-3} & \text{is represented by} & -b := z_1 z_3, \quad +a := z_2 z_3\ . \end{array} \tag{2.4}$$

My article [4] shows in fig. 1 the 2×3 trigons generated by an Euclidean triangle. Article [5] uses trigons in special Euclidean location, defined as pairs of vectors.

3. The composition of spherical particles

In general locations two tetraglobes T and T^* may be given by their corners $z_k \in \mathbb{C}_{\boldsymbol{i}}$ and $z_k^* \in \mathbb{C}_{\boldsymbol{i}^*}$, $k = 1, 2, 3, 4$. It is generally $\boldsymbol{i} \neq \boldsymbol{i}^*$.

I denote the angles α_k in the triangles of T as follows:

α_1 is the angle with the apexes z_2, z_4 and z_1, z_3

α_2 is the angle with the apexes z_3, z_2 and z_1, z_4

α_3 is the angle with the apexes z_1, z_2 and z_4, z_3.

(Angles of two 2-circles in conformal position are equal (Theorem II.2.2). The four triangles of a tetraglobe have angles of the same magnitude (Theorem II.2.3))

In the same way the three angles α_k^* of T^* may be denoted.

Both tetraglobes T and T^* are given a forward direction (Definition II.4.1). α_1 and α_1^* may be chosen as exponent angles. In this way both tetras are centred and directed, we get the spherical particles T_{+1} and T_{+1}^*.

Definition 3.1. *Two spherical particles T_{+1} and T_{+1}^* are in a* ***position of composition*** *if*

$$z_1 = z_1^*, \quad z_2 = z_4^*, \quad z_3 = z_3^* \; . \tag{3.1}$$

In general $z_4 \neq z_2^*$.

Theorem 3.1. *Two spherical particles T_{+1} and T_{+1}^*, generally located in the 3-dimensional space, can always be especially transformed in a position of composition.*

Proof. Always the conformal spheres $\mathbb{C}_{\boldsymbol{i}}$ and $\mathbb{C}_{\boldsymbol{i}^*}$ can be mapped by a conformal transformation of the 3-dimensional space in such a form that $\boldsymbol{i}$ and $\boldsymbol{i}^*$ are not changed but that z_3 is changed into z_3^*, $z_3 = z_3^*$. For instance an Euclidean translation can be used as this conformal transformation. It is always possible to describe the point $z_3 = z_3^*$ by the number ∞ so that $z_3 = z_3^* = \infty$. This point is chosen as the absolute point of Euclidean geometry so that the conformal spheres $\mathbb{C}_{\boldsymbol{i}}$ and $\mathbb{C}_{\boldsymbol{i}^*}$ are Euclidean planes.

We have to discuss the two possibilities $\boldsymbol{i} = \boldsymbol{i}^*$ and $\boldsymbol{i} \neq \boldsymbol{i}^*$:

(a) If $\boldsymbol{i} = \boldsymbol{i}^*$ $\mathbb{C}_{\boldsymbol{i}}$ and $\mathbb{C}_{\boldsymbol{i}^*}$ are parallel planes which will be identified. Because three points z_k of a conformal sphere can always be mapped by a conformal transformation in three other points z'_k of this sphere it is always possible to map the corners of the two trigons in such a way that $z_1 = z_1^*$, $z_2 = z_4^*$, $z_3 = z_3^*$. The position of composition is realised if $\boldsymbol{i} = \boldsymbol{i}^*$.

(b) If $\boldsymbol{i} \neq \boldsymbol{i}^*$ a conformal circle is the intersection of $\mathbb{C}_{\boldsymbol{i}}$ and $\mathbb{C}_{\boldsymbol{i}^*}$ which in the Euclidean location appears as a straight line. In the 2-dimensional spaces of both $\mathbb{C}_{\boldsymbol{i}}$ and $\mathbb{C}_{\boldsymbol{i}^*}$ always a conformal transformation exists which maps the corners of the trigons T_{+1} and T_{+1}^* in such a way that $z_1, z_2, z_3 \in \mathbb{C}_{\boldsymbol{i}}$ and $z_1^*, z_4^*, z_3^* \in \mathbb{C}_{\boldsymbol{i}^*}$ are points of the common straight line; and that $z_1 = z_1^*, z_2 = z_4^*, z_3 = z_3^*$. The position of composition is realised if $\boldsymbol{i} \neq \boldsymbol{i}^*$. □

The four points $z_1 = z_1^*, z_2^*, z_3 = z_3^*, z_4$ define a tetraglobe. Let T_p be one of its trigons. With $z_3 = \infty$ as absolute point this trigon shall be represented by the pair of vectors $x_1 := z_1 z_4$, $x_2 := z_1 z_2^*$, that means by the positive directed triangle $z_1 z_4 z_2^*$ with the centre z_1.

Definition 3.2. *The spherical particle T_p is the* ***composition*** *of the particles T_{+1} and T_{+1}^*, in signs*

$$T_p = T_{+1} \circ T_{+1}^* \,. \tag{3.2}$$

Remark

My article [5] started with centred and directed Euclidean triangles described by pairs of vectors, the specially Euclidean form of trigons [5, page 129]. In that article the above Definition 3.1 (position of composition) was also used. Theorem 2.1 [5, page 131] of that article was proved by starting with this special Euclidean concept of trigons.

Theorem 3.2. *Let A and B be two spherical particles and w_A, w_B the corresponding characteristic numbers of these trigons, respectively; let P be the composition of A and B. Then the quaternionic product $w_A \cdot w_B$ of these numbers w_A, w_B corresponds to the spherical particle P, that means this product $w_A \cdot w_B$ is the characteristic number of the trigon P.*

Proof. Without loss of generality this theorem can be proved by using an Euclidean composition position with $z_3 = z_3^* = \infty$. This special position is identical with the position of trigons used in [5] to prove Theorem 2.1 of that article [5, page 131]. Theorem 2.1 is the 'Euclidean' version of conformal Theorem 3.2. Because the Euclidean Theorem 2.1 is proved our conformal Theorem 3.2 is also proved by using a special Euclidean location and the traditional vector algebra. □

If $\boldsymbol{i} = \boldsymbol{i}^*$ Theorem 3.2 can also be proved directly and by using a composition position in general location:

The spherical particles A and B have the characteristic numbers (cross ratios)

$$w_A = (z_1 z_2 z_3 z_4) := (z_4 - z_1)(z_4 - z_3)^{-1}(z_2 - z_3)(z_2 - z_1)^{-1} \tag{3.3}$$

and

$$w_B = (z_1^* z_2^* z_3^* z_4^*) := (z_4^* - z_1^*)(z_4^* - z_3^*)^{-1}(z_2^* - z_3^*)(z_2^* - z_1^*)^{-1} . \tag{3.4}$$

With $z_1 = z_1^*, z_2 = z_4^*, z_3 = z_3^*$ (position of composition) it follows

$$w_A \cdot w_B = (z_4 - z_1)(z_4 - z_3)^{-1}(z_2^* - z_3)(z_2^* - z_1)^{-1} . \tag{3.5}$$

This is the characteristic number (cross ratio)

$$w_P := (z_1 z_2^* z_3 z_4) = (z_1^* z_2^* z_3^* z_4) \tag{3.6}$$

of P. The multiplication

$$w_P = w_A \cdot w_B \tag{3.7}$$

of the characteristic numbers corresponds to the composition of the spherical particles

$$P = A \circ B \; . \tag{3.8}$$

□

The isomorphism proved in Theorem 3.2 leads to an identification of a 'quaternion' with a 'spherical particle' ('directed and centred tetraglobe', 'trigon'): A quaternion is the algebraic form of a spherical particle. A spherical particle is the geometrical form of a quaternion. Numbers, elements of the quaternionic skew field, are spherical particles of the 3-dimensional conformal space.

4. Complex numbers as basic elements of natural space

With the aim to define and construct conformal figures we used elements of the complex field to describe points. Here 'point' did not have an invariant meaning, for every point could be transformed with the help of conformal transformations to any other point. Only 2-circles [6] and 4-circles [7] had objective (invariant) numerical qualities. The real characteristic number of a 2-circle and the complex characteristic number of a 4-circle describe these objective qualities. *On this level the using of numbers acquires a new quality*: An individual number describes in an invariant form the individual shape of a geometrical figure. In using directed and centred 4-circles we get a one-to-one correspondence between an individual figure and an individual number. In this form a trigon is the model of a number and the number the algebraic substratum of a geometrical figure. In this sense we may say: Numbers as spherical particles (not as points) are the basic elements of natural space. These first elements of space possess a very characteristic quality: They have a well defined shape but not a defined extent ('grösse') because they are only figures with angles not with lengths.

Remark

The fondly written book of P.J. Nahin [3] tells the story of the imaginary unit, geometrically interpreted as point of a plane and as an operator in this plane by Argand, Buée, Gauss, Wessel. My new, more general interpretation of $\boldsymbol{i}$ as an $\boldsymbol{i}$-trigon identifies this unit with a geometrical *figure*, with a spherical particle of a special shape. In Euclidean location this figure is an *isosceles right-angled triangle.*

In the following we will give the description of points an objective meaning. On this way we also come back to the traditional geometrical interpretation of $\boldsymbol{i}$. At first we discuss only the situation in two dimensions. *So far we have not used coordinate systems.* A spherical particle with the characteristic number $\boldsymbol{i}$ can be used as a Cartesian coordinate system if this $\boldsymbol{i}$-trigon possesses a unit of length.

Definition 4.1. *The four points z_k of an $\boldsymbol{i}$-trigon define a* ***Cartesian coordinate system*** *S with the* ***origin*** *z_1, the* ***absolute Euclidean point*** *z_3 and the* ***unit point*** *z_2. The circle $z_1z_2z_3$ is the* ***first axis*** *of the Cartesian system.*

In using a Cartesian system *with an absolute point* we go back from the conformal to the Euclidean level.

Definition 4.2. *Origin and absolute and unit points define the* ***coordinate*** *w of a point z relative to S*

$$w := (z_1z_2z_3z) := (z - z_1)(z - z_3)^{-1}(z_2 - z_3)(z_2 - z_1)^{-1}. \tag{4.1}$$

With this definition the coordinate w of a point **relative to a coordinate system** is an objective quality of the point, for the coordinate is defined as a cross ratio which does not change in the Möbius group $\mathbb{M}$ if the coordinate system together with the point is mapped.

If absolute point and origin are especially described by $z_3 = \infty$ and $z_1 = 0$ the coordinate of the point z is

$$w := z/z_2 \,. \tag{4.2}$$

If the Cartesian system possesses a unit of length this unit can be introduced by choosing a number for z_2. If 'second' is the unit of length with $z_2 = 1$ for instance the point $z = 3$ on the first axis has a distance of 3 seconds from the origin. In a naturally used coordinate system distances in space and time should be measured in the same unit. I think the points on the first axis as points in time; points on the second axis $z_1 z_3 z_4$ are points in space. The point $z_4 = \boldsymbol{i}$ of the Cartesian system has the coordinate $w = \boldsymbol{i}/z_2$. 'The point $3\boldsymbol{i}$ on the space axis has a distance of 3 seconds from the origin' means that light travels this distance of space in 3 seconds if the velocity of light is measured by the number one.

If space is reduced to one dimension the description of events with the help of coordinates can use ordinary complex numbers. For example the coordinate

$$w = 3 + 5\boldsymbol{i} \tag{4.3}$$

describes a kinematic physical event with a time distance of 3 and a space distance of 5 seconds from the origin.

In the same form a dynamic event can be described, if the dynamic unit, for example, 'kilogram' is used both for measuring masses and impulses.

Remark

If the imaginary unit i is defined as a pair of real numbers $i := (0, 1)$ this complex number is traditionally often seen as a vector ('zeiger', pointer) in the plane $z_1 z_2 z_4$. I substituted i by $\boldsymbol{i}$ (compare Section II.4, Remark 3) to see elements of $\mathbb{C}$ as elements of $\mathbb{H}$ (Hamiltonian quaternions as defined in [5]). $\boldsymbol{i}$ is a quaternionic vector, right-angled to the plane $z_1 z_2 z_4$. In relation to the Cartesian system S the point z_4 can be described either by the vector i or the trigon $w = \boldsymbol{i}$. For measuring lengths i and $\boldsymbol{i}$ can be identified. - After this identification we may see the Gauss plane realised by both the vector pair (e, i) and the vector pair $(e, \boldsymbol{i})$ $(e := z_1 z_2)$.

Complex numbers in their simple commutative form reflect the essential components of physical events: On the conformal level the elements of this number field describe the numerical structure of events in a very pure form. These elements conformally appear in their exponential form

$$w = \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot \exp(\boldsymbol{i}\alpha_1), \quad \sum \alpha_k = \pi \; . \tag{4.4}$$

On the conformal level a spherical particle is a pure angle figure ('trigon') swimming in the 3-dimensional natural space. In the natural conformal space this particle exists *without a 'time-space separation'*

$$w = sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot (\cos\alpha_1 + \boldsymbol{i} \cdot \sin\alpha_1) = t + x = \xi_1 + \boldsymbol{i} \cdot \xi_2 \; . \tag{4.5}$$

In using a Cartesian coordinate system and its unit of length a physical event gets a characteristic feature by the separation of its quaternion in real and vector part. Now physical events are kinematically described by

$$Q = t + x \tag{4.6}$$

and dynamically by

$$P = m + p \; . \tag{4.7}$$

Remark

My article [5, pages 133-134] explained that we have to measure space in *imaginary* units with $h = i$ if we want to describe space-time and material events with the help of quaternions. More generally we have this situation: Physical events can be described either by using the metrical fundamentals $(c, h) = (1, i)$ or $(c, h) = (i, 1)$. With this physical use of Hamiltonian complex numbers time dimension (and mass dimension) of an event is described by the real part, space dimensions (and impulse dimensions) are described by the vector part of complex numbers. And every vector part x (or p) of a complex number has the quality x^2 (or p^2) ≤ 0.

5. Quaternions as basic particles in the natural space

Now we discuss the situation in 3 and 4 dimensions.

Quaternions seen as spherical particles are entities 'only in three dimensions'. One may say: Quaternions as elements of the 3-dimensional conformal space possess two dimensions in their sphere $\mathbb{C}_{\boldsymbol{i}}$ and two dimensions by moving (changing) the position $\boldsymbol{i}$ of their sphere in the natural space. A quaternion

$$A = \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot \exp(\boldsymbol{i}\alpha_1) \tag{5.1}$$

with its three trigon angles α_k, its angle sum $\sum \alpha_k = \pi$ and its position $\boldsymbol{i}$ is on the conformal level a pure angle entity, too. On this level the first particles of the natural space possess a complete Euclidean trigonometry [7] but without units of lengths. On the conformal level and in an exact sense we cannot say that a natural spherical particle is 'small' or 'big'. This particle has a fixed position $\boldsymbol{i}$ in the 3-dimensional space but its diameter and its location (locality) is not defined.

The 'conformal' exponential form of a quaternion can generally be written also in the additive form

$$A = \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot (\cos\alpha_1 + \boldsymbol{i}\cdot\sin\alpha_1) = t + x \quad (\text{or } m + p) \tag{5.2}$$

of real and vector part. Also on the conformal level we get a geometrical meaning of both parts. For example in

$$A = \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot \exp(\boldsymbol{i}\alpha_1) = \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot (\cos\alpha_1 + \boldsymbol{i}\cdot\sin\alpha_1) \tag{5.3}$$

with

$$t = \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot \cos\alpha_1 \quad \text{and} \quad x = \boldsymbol{i} \cdot \sin^{-1}\alpha_3 \cdot \sin\alpha_2 \cdot \sin\alpha_1, \quad \sum \alpha_k = \pi, \quad (5.4)$$

the vector part x of the number describes especially an ortho-tetra if $t = 0$:

$$x = \boldsymbol{i} \cdot \tan\alpha_2 \qquad (5.5)$$

(compare (II.6.2)).

On the conformal level only this angle form of x has an objective meaning. x describes as a special cross ratio a conformal *right-angled* triangle but we cannot use this vector part as a 'point vector', as a pointer to the space part of a kinematic event without further assumptions.

At first we have to choose a Cartesian coordinate system together with a unit of length. Also in this more general case we choose an $\boldsymbol{i}$-trigon as Cartesian system. The third axis is defined by the position of $\boldsymbol{i}$. All three coordinate axes represent a spatial dimension. The geometrical and physical meaning of the additive form of a number $A = t + x$ is completely revealed if we use a three dimensional Cartesian system and the unit of length in this system to describe points in the natural 3-dimensional space with help of the quaternionic vectors x together with the time dimension t as the real part of this quaternionic number. We may also see this real part as a 'fourth dimension' but because our visual perception is restricted to three dimensions we often have to go back to the Gauss plane where the 3-dimensional natural space is restricted to the 1-dimensional 'imaginary axis'. We cannot visually perceive a $(1 + 3)$-dimensional Gauss plane. But the traditional Gauss plane can illustrate that it is useful to perceive the 3-dimensional space of our visual perception as a *3-dimensional 'imaginary axis'* described with the help of a Cartesian system and *by vectors satisfying* $x^2 \leq 0$.

In the past physicists explained a physical phenomenon with the help of mechanics. A progress was made with also accepting, for example, electromagnetic structures as fundamental physical structures. Because geometrical and physical basic structures possess a conformal background with a nearly complete Euclidean trigonometry, science should also ask which physical structures can be understood only with the help of angle structures and the measurement of angles, without separating geometrical (physical) particles in their time and space components.

Conformal tetraglobes and therefore Hamiltonian complex numbers possess a quality which is not only an invariant of the conformal group, but this general invariant does not change with transformations which map a complex number into a complex number. This quality has a logarithmic and a numerical form and appears in traditional geometry as the sum of angles of an Euclidean triangle.

Definition 5.1. *The product of a positive or negative cycle of tetraglobic numbers* w_{+k} *or* w_{-k} *is the* ***charge*** ε *of a tetraglobe:*

$$\varepsilon := w_{+1} \cdot w_{+2} \cdot w_{+3} = w_{-1} \cdot w_{-2} \cdot w_{-3} = \mathbf{-1} \tag{5.6}$$

(Compare Equation (I.5.4))

Every tetraglobe possesses this charge $\varepsilon = \mathbf{-1}$. The characteristic numbers w_{+k} or w_{-k} of a tetraglobe my be changed by a transformation f into other numbers $w^*_{\pm k} = f(w_{\pm k})$ but it is again

$$\varepsilon^* := w^*_{+1} \cdot w^*_{+2} \cdot w^*_{+3} = w^*_{-1} \cdot w^*_{-2} \cdot w^*_{-3} = \mathbf{-1}\ . \tag{5.7}$$

Remarks

1. I risk to denote this general invariant of a tetraglobe as 'charge'. In a natural metrical system also the electronic charge is measured by the number **-1**.
2. Not only tetraglobes but all conformal 2^ν-circles ($\nu = 0, 1, 2, 3$) possess the number '**-1**' as an invariant number.

Definition 5.2. *The logarithm of the tetraglobic charge is the* ***spin*** σ_n *of a tetraglobe,*

$$2\pi \cdot \sigma_n := \ln(\varepsilon) \ . \tag{5.8}$$

Because $\ln(-1) = \boldsymbol{i}\pi \cdot (1 + 2n), n \in \mathbb{N}_0$, *it is*

$$\sigma_n = \boldsymbol{i} \cdot (\tfrac{1}{2} + n) \ . \tag{5.9}$$

The two spin *directions* of a tetraglobe were defined in Definition II.4.3 with the help of its position $\boldsymbol{i}$. Definition 5.2 leads to the natural spin *scale* of quaternions and gives a natural explanation of the number $\frac{1}{2}$ in $\sigma_0 = \frac{1}{2}\boldsymbol{i}$.

The difference of the quaternionic $\boldsymbol{i}$ and the 'scalar' unit i was emphasised in Section II.11. I define this fundamental unit i, also geometrically but without using any special metrical level, with the help of the tetraglobic charge:

Definition 5.3. *The* ***scalar imaginary unit*** i *is the square root of the tetraglobic charge*

$$i := \sqrt{\varepsilon} \ . \tag{5.10}$$

The Definition 5.1 together with the Definitions 5.2 and 5.3 underlines that the charge ε, together with the connected units i and σ_n are invariants, objective qualities of any number not only on the Euclidean or conformal but on an extremely general level.

6. Concluding remarks

Hamilton's numbers are Euclid's triangles, centred and directed. The young Feynman was wondering about the mathematical formula $-1 = e^{\boldsymbol{i}\pi}$ ("The most remarkable formula in math." [2, page 35]). Is my *geometrical* connection of charge, spin, angle sum in Euclidean triangles and position unit i of numbers, described by $-1 = e^{i\pi}$, also a *physical* connection? Do the Euclidean angles α_k and their characteristic numbers w_k constitute a tetraglobe as *quarks* constitute a physical particle?

Is an Einsteinean 'field theory' still possible? This question is connected with the problem if a field theory can accept and describe non-locality. The 'tetraglobic field theory' is conformally - free of Cartesian coordinates - a theory with particles, not with points (or vectors) as basic elements. This particles possess not a location in an objective sense. I have not discussed in detail if for example a 'chain of tetraglobes' (tetras connected by an angle) can be used to transport information or energy. Such transfer, if possible, is a transport not in space-time but in the conformal background of space-time, not bound by the velocity of light.

My form of describing space-time and the dynamic aspect of physical events by

$$Q = t + x \qquad \text{and} \qquad P = m + p$$

underlines the isomorphism of both aspects. Classical physics sees both aspects 'instantly' connected, quantum physics emphasises its alternative nature. It is known that physicists have to *decide* for measuring the kinematic aspect $Q = (t, x)$ or the dynamic aspect $P = (m, p)$ of an event if they want exact results (Heisenberg). Physics uses the correlation of both aspects (for instance Dirac [1, page 110]: "The theory of relativity puts energy in the same relation to time as momentum to distance") but I do not know a theory which *explains* this correspondence as be based on more fundamental principles. I have the idea that one conformal particle is bound with both aspects. That the possibility to describe the conformal trigonometry of tetraglobes with real *or* with imaginary units of angles ([7], Section 11); that Hamiltonian quaternions can be described *both* with the fundamentals $(c, h) = (i, 1)$ and the fundamentals $(c, h) = (1, i)$ [5] can help us to understand the 'complementary' (isomorphic) aspects of space-time and matter. Until this day

I had neither the time nor the energy for a detailed formulation of the idea that both length metrical aspects Q and P of the physical world are connected with the same entity in the conformal background of this world.

Now I can see this physical world created as a conformal circle (or a projective number line) with Hamiltonian complex numbers as its elements. Our visual perception realises these elements as spherical particles in three dimensions. Only on the metrical level of lengths events can be seen as points of a world with 1 + 3 dimensions, described by Cartesian systems. The quaternionic skew field is the only topological field which is connected and locally compact (L.S. Pontrjagin); which possesses a 'normal' infinitesimal calculus. Mathematics possesses one and only one system of numbers. One and only one physical world is created which human beings can describe and understand with the help of this number system. Science makes a methodical progress if some parts of theoretical physics are only seen as the complex function theory on the set of Hamilton's numbers?

The idea to describe geometry and physics with the help of numbers was introduced by Descartes. I am taking a further step in the same direction: Not only real numbers, not only vector algebra but Hamiltonian complex numbers can help to realise this Cartesian programme.

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